The Koch "Snowflake" Curve

W. L. Culbertson

One of the first fractal curves to be examined, the Koch snowflake (or the Koch star or the Koch island) was described in a 1904 paper by the Swedish mathematician Helge von Koch. He noted it as "a continuous curve without tangents." This curve has an infinite length, but a finite area.

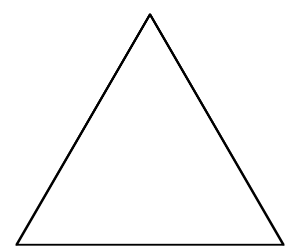
A "curve" in this case means a continuous series of points which may not all be in the same direction. When we think of mathematical curves, we tend to think of circles or parabolas or some such figure. However, a triangle is a type of curve. It just has a sharper change in direction for the line than we are used to thinking of. (For a more abstract investigation of the properties of curves and shapes, see the area of mathematics called topology.)

Let's construct one of Koch's curves and investigate those ideas about "infinite length" and "finite area" mathematically.

Step One:

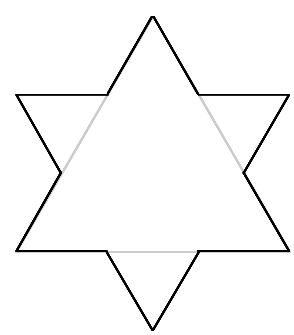
To construct a Koch Snowflake, begin by drawing an equilateral triangle. This triangle has three equal sides, and we can represent their length as some number *a*. The length of this first figure, our first "curve," is just the perimeter of the triangle, P_1, which would be the three sides added together:

$$P_1 = 3a$$



Second Iteration:

Now divide each of the sides into thirds and construct new equilateral triangles on the centers of each of the sides. The new shape is a six-pointed "star."

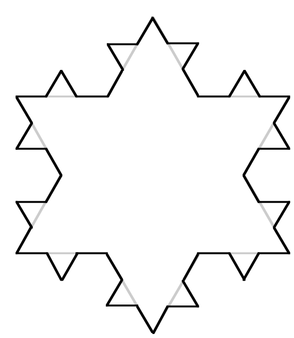


Now there are 12 sides. On each of the three original sides, there are now four segments, each $\frac{1}{3}a$ long. The new perimeter is

$$P_2 = 3 \cdot 4 \cdot \frac{1}{3}a = 3 \cdot \frac{4}{3}a$$

Third Iteration:

Once again divide each of the twelve sides into thirds and put smaller equilateral triangles on each of those sides. Although it gets harder and harder to draw the little triangles, you can imagine continuing the process on and



on with smaller and smaller triangles. (In Wikipedia, the entry under "Koch Snowflake" shows an animation of the first seven iterations of the process—fascinating to watch!)

So how many sides are there? Each of the three original sides sprouted four segments, so there were $3\times 4=12$ sides at that step. In the third step, each of the 3×4 sides have four divisions, so there would be a total of $3\times 4\times 4$ sides (that's 48 if you are keeping score) in this new figure.

Each of these new sides in the third step would be a third of the size of a side in step 2 or $\frac{1}{3}\left(\frac{1}{3}a\right)$. Therefore, the perimeter of the curve after three iterations would be the number of sides times the length of a side or

$$P_3=(3\cdot 4\cdot 4)\cdot\left(\frac{1}{3}\cdot\frac{1}{3}a\right)=3\cdot\frac{4}{3}\cdot\frac{4}{3}\cdot a \quad \text{or}$$

$$P_3=3\left(\frac{4}{3}\right)^2 a$$

Fourth Iteration:

When we repeat the process a fourth time, our snowflake is getting positively frilly. By similar arguments, each of the previous sides adds four segments, and each of the segments is a third the length of the previous side.

$$P_4=(3\cdot 4\cdot 4\cdot 4)\cdot\left(\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}a\right)=3\cdot\frac{4}{3}\cdot\frac{4}{3}\cdot\frac{4}{3}\cdot a$$

$$P_4=3\left(\frac{4}{3}\right)^3 a$$

n^th Iteration:

If we continue this process some arbitrary number, *n*, times, you should see that we would have $3\cdot 4^{n-1}$ sides with each side having length $\left(\frac{1}{3}\right)^{n-1}a$. The perimeter is

$$P_n=3\left(\frac{4}{3}\right)^{n-1} a$$

The perimeter is in the form of a geometric sequence which has the form ar^n. Since the number we are taking to the power, $\frac{4}{3}>1$, the perimeter keeps growing each time we do another iteration of adding triangles. The perimeter will grow infinitely long. Specifically, the limit of the perimeter with an infinite number of sides would be:

$$\lim_{n\to\infty}P_n=\lim_{n\to\infty}3\left(\frac{4}{3}\right)^n a=\infty$$

Area of the Koch Curve:

What about the area of the Koch Curve? Just by inspection we can see that the area must be finite. Draw a circle which circumscribes the original equilateral triangle. That would be a circle centered on the triangle which passes through all three vertices.

Where would that center of the triangle be? If we draw the bisectors of the angles of the equilateral triangle, they all meet at a common point — the center of the triangle. Those lines are also perpendicular to the opposite sides, so they are also a height of the triangle. If the sides of the triangle have length, *a*, from geometry we know that in an equilateral triangle, $\text{height} = \frac{a\sqrt{3}}{2}$. Looking at the other 30°-60°-90° triangles in the diagram, we can figure of the distance from the center to a vertex of the triangle is $\frac{a\sqrt{3}}{3}$. That would be the radius of the circumscribing circle.

In the second step of drawing the Koch curve, we add three new triangles on the sides of the original sides. These new triangles have sides one-third the sides of the original and their heights would be one-third as tall or

$$\frac{1}{3}\left(\frac{a\sqrt{3}}{2}\right) = \frac{a\sqrt{3}}{6}.$$

We add this height to the center of one of the sides of the original triangle. That center of one of the sides of a triangle is $\frac{a\sqrt{3}}{6}$ away from the center of the triangle, so the point of the added triangle is at a distance of $\frac{a\sqrt{3}}{6} + \frac{a\sqrt{3}}{6} = \frac{a\sqrt{3}}{3}$.

In other words, the points of our new triangles are at the same distance from the center as the original vertices of the triangle. New triangles after this step will be added to those sides and so on never getting any farther away from the center than those initial vertices.

If the Koch Curve is contained within a circle of radius $\frac{a\sqrt{3}}{3}$, then we can calculate the area of the enclosing circle as:

$$A_{\text{circle}} = \pi r^2 = \pi\left(\frac{a\sqrt{3}}{3}\right) \quad \text{so the area of our curve,} \quad A_{\text{Koch Curve}} < \frac{\pi\sqrt{3}}{3}\cdot a$$

But, we can do better than that! It is possible to calculate the exact area of this curve with an infinitely long boundary. Take the original equilateral triangle with side of length *a*. The familiar formula for the are of a triangle is $\text{Area} = \frac{1}{2} \times (\text{base}) \times (\text{height})$.

For our triangle, the base = *a* and $\text{height} = \dfrac{a\sqrt{3}}{2}$. Therefore, the area of our base triangle is $A_0 = \dfrac{1}{2} \cdot (a) \cdot \left(\dfrac{a\sqrt{3}}{2} \right)$

✪ or $A_0 = \dfrac{a^2\sqrt{3}}{4}$ (The ✪ means we are going to need to use this again.)

First Iteration:

For the next stage, add three equilateral triangles, each side one-third the length of a side of the original triangle, in the middle of each side of the original triangle. How much area, A_1, did we add? Since each side of the new triangles is $\dfrac{1}{3} \cdot a$ long, the height would be: $h = \left(\dfrac{1}{3} \right) \cdot \dfrac{a\sqrt{3}}{2}$. Therefore, the area of each of these triangles would be:

$$A = \frac{1}{2} \cdot \left(a \cdot \frac{1}{3} \right) \left(\frac{1}{3} \cdot \frac{a\sqrt{3}}{2} \right)$$

Rearranging the factors into more convenient form gives $A = \left(\dfrac{a^2\sqrt{3}}{4} \right) \left(\dfrac{1}{9} \right)$. Why is that more convenient? Remember that our original triangle had an area $A_0 = \dfrac{a^2\sqrt{3}}{4}$, so the area of our added triangles are each $A = \left(\dfrac{1}{9} \right) A_0$.

Since we are grafting on three triangles, the area added in the first iteration is

$$A_1 = 3 \cdot \left(\frac{1}{9} \right) A_0 = \left(\frac{1}{3} \right) A_0 .$$

Second Iteration:

For the second addition to the Koch Curve, add 3×4 triangles with sides $\dfrac{1}{3} \cdot \left(\dfrac{1}{3} a \right) = \dfrac{1}{9} a$ and height $h = \left(\dfrac{1}{9} \right) \cdot \dfrac{a\sqrt{3}}{2}$

The area of each triangle would be: $A = \dfrac{1}{2} \cdot \left(\dfrac{1}{9} a \right) \left(\dfrac{1}{9} \cdot \dfrac{a\sqrt{3}}{2} \right)$,

and rearranging to get the original area again: $= \left(\dfrac{a^2\sqrt{3}}{4} \right) \left(\dfrac{1}{9} \right)^2 = \left(\dfrac{1}{9} \right)^2 A_0$

We have 3×4 of these new triangles, so the total area added is $A_2 = 3 \cdot 4 \cdot \left(\dfrac{1}{9} \right)^2 \cdot A_0$.

We need to multiply this out and rearrange to get a pattern going. The area added in the second iteration is: $A_2 = \frac{1}{3} \cdot \left(\frac{4}{9}\right) \cdot A_0$.

Third Iteration:

The third time, our added triangles have sides $\frac{1}{3} \cdot \left(\frac{1}{9} a\right)$, the height would be $h = \frac{1}{3}\left(\frac{1}{9} \cdot \frac{a\sqrt{3}}{2}\right)$, and the area of one triangle would be

$$A = \frac{1}{2} \cdot \left(\frac{1}{3} \cdot \left(\frac{1}{9} a\right)\right)\left(\frac{1}{3}\left(\frac{1}{9} \cdot \frac{a\sqrt{3}}{2}\right)\right).$$

With some rearranging of the factors as before, we get

$$A = \left(\frac{1}{9}\right)^3 A_0 .$$

We are adding $3 \times 4 \times 4$ of them. hopefully you see the pattern emerging. This time we will add a new area

$$A_3 = 3 \cdot 4 \cdot 4 \cdot \left(\frac{1}{9}\right)^3 A_0 = \frac{1}{3} \cdot \left(\frac{4}{9}\right)^2 \cdot A_0$$

n[th] Iteration:

If you look at the number of repeated multiplications developing, we can write a general expression for the area added in the *n*[th] iteration, A_n.

$$A_n = 3 \cdot 4^{(n-1)} \cdot \left(\frac{1}{9}\right)^n A_0 = 3 \cdot 4^{(n-1)} \cdot \frac{1}{9} \cdot \left(\frac{1}{9}\right)^{n-1} A_0$$

or $$A_n = \frac{1}{3} \cdot \left(\frac{4}{9}\right)^{n-1} A_0$$

Total Area:

$$\sum_{n=0}^{\infty} A_n = A_0 + A_1 + A_2 + \cdots$$

$$\sum_{n=0}^{\infty} A_n = A_0 + \frac{1}{3} \cdot A_0 + \frac{1}{3} \cdot \left(\frac{4}{9}\right) \cdot A_0 + \frac{1}{3} \cdot \left(\frac{4}{9}\right)^2 \cdot A_0 + \frac{1}{3} \cdot \left(\frac{4}{9}\right)^3 \cdot A_0 + \cdots$$

Factor out the $\frac{1}{3} A_0$ common factor starting in term 2.

$$\sum_{n=0}^{\infty} A_n = A_0 + \frac{1}{3}A_0\left(1+\left(\frac{4}{9}\right)+\left(\frac{4}{9}\right)^2+\left(\frac{4}{9}\right)^3+\cdots\right)$$

Writing the quantity in parentheses as a sum gives:

$$\sum_{n=0}^{\infty} A_n = A_0 + \frac{1}{3}A_0\sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n$$

The summation is a geometric series of the form:

$\sum_{n=0}^{\infty} a\cdot r^n$ and since $r<1$ the sum is $\sum_{n=0}^{\infty} a\cdot r^n = \frac{a}{1-r}$

(This result is from previous work.)

So I can write the area of a Koch Curve as:

$$\sum_{n=0}^{\infty} A_n = A_0 + \frac{1}{3}A_0\sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n = A_0 + \frac{1}{3}A_0\left(\frac{1}{1-\frac{4}{9}}\right)$$

$$= A_0 + \frac{1}{3}A_0\left(\frac{1}{\frac{5}{9}}\right) = A_0 + \frac{1}{3}A_0\left(\frac{9}{5}\right) = A_0 + \frac{3}{5}A_0$$

$$= \frac{8}{5}A_0$$

What did Koch mean when he said this curve had no tangents? When we think about a line tangent to a curve, we think of "a line that just touches the line at one point." Hmm, not good enough. What if the line crosses through the curve? That would be a "touch" at one point.

A better idea is to think about a tangent line is as a line that is "going the same direction" as the curve at that particular point. But what does that mean in mathematical terms?

Mathematicians define a tangent line in terms of a limit process. Since we can't define a line with a single point (our point of tangency), we can use a point a small distance from our tangent point to construct a secant line connecting the two points on the curve. The closer our second point is to the point of tangency, the better the agreement will be with the actual direction of the tangent line. But between those two points, there would be an infinite number of those added little triangles — there is no way to take the limit!