Notes on Integration By Parts

Using the Product Rule for differentiation allows us to derive [see your notes or the text] a method for integrating certain types of integrals involving a product. The formula for Integration by Parts is:

$$\int u \cdot dv = uv - \int v \cdot du$$

We have a product (left side) to integrate with no easy way to do it. The formula allows us to produce an integral on the right side, $\int v \cdot du$, which is easier to integrate. You get to pick the u and the dv from the given integral. Things to look for:

1. Choose the dv as the most complicated part of the integral that has a <u>simple</u> integral.

Example: Find $\int x \cdot \ln x \, dx$.

If we choose $dv = \ln x \, dx$, we are stuck with an ugly integral to find v.

$u = x$	—>	$du = dx$
$v = !!$	<—	$dv = \ln x \, dx$

However, if we choose $u = \ln x$, things look a lot better:

$u = \ln x$	—>	$du = \frac{1}{x} dx$
$v = \frac{1}{2}x^2$	<—	$dv = x \, dx$

Using integration by parts:

$$\int x \cdot \ln x \, dx = (\ln x)\left(\tfrac{1}{2}x^2\right) - \int \tfrac{1}{2}x^2 \cdot \frac{1}{x} \, dx$$

$$= (\ln x)\left(\tfrac{1}{2}x^2\right) - \int \tfrac{1}{2}x \, dx$$

Notice how the situation has changed? That new integral on the right is easy, and the answer would be:

$$\int x \cdot \ln x \, dx = (\ln x)\left(\tfrac{1}{2}x^2\right) - \tfrac{1}{4}x^2 + C$$

Or, factoring to give:

$$= \tfrac{1}{4}x^2(2\ln x - 1) + C$$

2. Choose the u part as an expression that will simplify when it is differentiated.

Example: Find $\int x^2 \cdot e^x \, dx$.

If we choose $u = e^x$, nothing much would change if we took the derivative. (On the other hand, integrating it doesn't change much either. The function e^x is agnostic with respect to either operation.) If $u = x^2$, then $du = 2x \, dx$, which looks like progress. Let:

$u = x^2$	—>	$du = 2x \, dx$
$v = e^x$	<—	$dv = e^x \, dx$

Now using integration by parts:

$$\int x^2 \cdot e^x \, dx = x^2 \cdot e^x - \int 2x \cdot e^x \, dx$$

Examining the new integral, you can see a dastardly little trick — another product! However, this product has a term with a smaller degree than the first one. Another integration by parts is necessary for this new integral.

For the new integral $\int 2x \cdot e^x \, dx$

$u = 2x$	—>	$du = 2 \, dx$
$v = e^x$	<—	$dv = e^x \, dx$

Which gives:

$$\int 2x \cdot e^x \, dx = 2x \cdot e^x - \int 2 \cdot e^x \, dx$$

Or, putting that into the context of the original problems:

$$\begin{aligned} \int x^2 \cdot e^x \, dx &= x^2 \cdot e^x - \left[2x \cdot e^x - \int 2 \cdot e^x \, dx \right] \\ &= x^2 \cdot e^x - 2x \cdot e^x + 2 \int e^x \, dx \\ &= x^2 \cdot e^x - 2x \cdot e^x + 2e^x + C \end{aligned}$$

Or, factoring out the e^x, we can see a product with a polynomial:

$$= e^x \left(x^2 - 2x + 2 \right) + C$$

3. Sometimes the original integral reappears in the process of integrating. Oddly enough, instead of being a problem, that can be the ticket to a solution.

Example: Find $\int e^x \cdot \sin x \, dx$.

Neither one of those factors will vanish if we differentiate (as in the previous example). But, plunge blindly ahead choosing $u = \sin x$ and $dv = e^x \, dx$:

$$u = \sin x \quad \longrightarrow \quad du = \cos x \, dx$$
$$v = e^x \quad \longleftarrow \quad dv = e^x \, dx$$

Integrating by parts, we get another product integral on the right side:

$$\int e^x \cdot \sin x \, dx = e^x \sin x - \int e^x \cdot \cos x \, dx$$

To integrate the new product integral on the right side, $\int e^x \cdot \cos x \, dx$, choose:

$$u = \cos x \quad \longrightarrow \quad du = -\sin x \, dx$$
$$v = e^x \quad \longleftarrow \quad dv = e^x \, dx$$

Which gives:

$$\int e^x \cdot \cos x \, dx = e^x \cos x - \int e^x \cdot (-\sin x) \, dx$$

Or, putting that into the context of the original problems:

$$\begin{aligned} \int e^x \cdot \sin x \, dx &= e^x \sin x - \left[e^x \cos x - \int e^x \cdot (-\sin x) \, dx \right] \\ &= e^x \sin x - e^x \cos x - \int e^x \cdot \sin x \, dx \end{aligned}$$

While in the depths of despair over having yet another product integral to deal with on the right-hand side, you may notice that the right hand product integral is identical to the original problem! Strange as it may seem at first, those are *like terms*, and you can collect them together on the left side:

$$2 \cdot \int e^x \cdot \sin x \, dx = e^x \sin x - e^x \cos x$$

Now you can get rid of the two by multiplying by $\frac{1}{2}$ on both sides to give:

$$\int e^x \cdot \sin x \, dx = \tfrac{1}{2}\left(e^x \sin x - e^x \cos x \right) + C$$

Factoring out an e^x gives the answer in the form of a product:

$$\int e^x \cdot \sin x \, dx = \frac{e^x}{2}(\sin x - \cos x) + C$$