Change in Angle of a Falling Body General Solution

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An object dropped from a height, s_0, falls freely under the acceleration of gravity. The general formula for distance above the ground of a falling body (units: feet & seconds) is

$$s = -16t^2 + V_0 t + s_0 .$$

The object will fall from the release time, $t = 0$, (note this means $V_0 = 0$) until the object hits the ground at $s = 0$ or

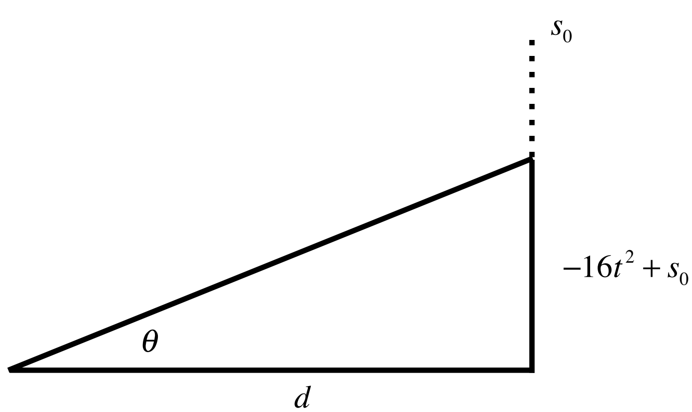
$$0 = -16t^2 + s_0$$

a) How long does it take to hit the ground?

$$16t^2 = s_0$$

$$t = \tfrac{1}{4}\sqrt{s_0}$$

b) What is the rate of change in viewing angle θ for an observer at distance *d*? (for $0 < t < \tfrac{1}{4}\sqrt{s_0}$)



The height of the object at time = *t* is given by $-16t^2 + s_0$. To get an angle to work with, find a basic trig ratio. At distance = *d*, the tangent ratio would be

$$\tan\theta = \frac{-16t^2 + s_0}{d} .$$

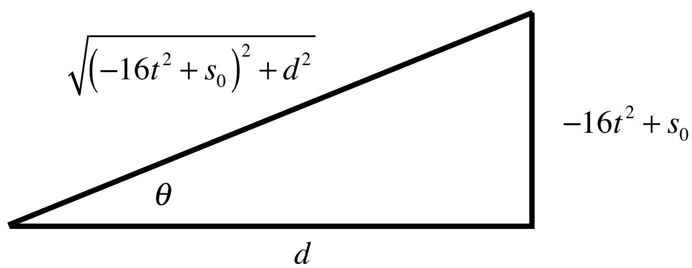
To find the rate of change of the angle, $\dfrac{d\theta}{dt}$, differentiate both sides implicitly to get

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{32}{d} t$$

Now solve for $\frac{d\theta}{dt}$

$$\frac{d\theta}{dt} = -\frac{32}{d}t \cdot \frac{1}{\sec^2\theta}\text{ , or}$$
$$\frac{d\theta}{dt} = -\frac{32}{d}t\cos^2\theta$$

This expression for the rate of change of the angle θ contains the angle itself. We need to get an expression in t, so we must replace the $\cos\theta$ term with the cosine ratio from the diagram. The cosine ratio must be the adjacent side (d) over the hypotenuse. To find an expression for the hypotenuse, we must use the Pythagorean Theorem.



Since $\cos\theta = \frac{\text{adj.}}{\text{hyp.}}$, for our angle:

$$\cos\theta = \frac{d}{\sqrt{\left(-16t^2 + s_0\right)^2 + d^2}}$$

The expression we are substituting into contains a $\cos^2\theta$ term:

$$\cos^2\theta = \frac{d^2}{\left(-16t^2 + s_0\right)^2 + d^2} = \frac{d^2}{256t^4 - 32s_0t^2 + s_0{}^2 + d^2}$$

Now we can substitute:

$$\frac{d\theta}{dt} = -\frac{32}{d}t\cos^2\theta = -\frac{32}{d}t \cdot \frac{d^2}{256t^4 - 32s_0t^2 + s_0{}^2 + d^2}$$

Therefore, the rate of change in angle is:

$$\frac{d\theta}{dt} = -\frac{32d^2t}{256dt^4 - 32ds_0t^2 + ds_0^2 + d^3}$$