A Brief History of Calculus

William L. Culbertson, Ph.D

Calculus allows mathematicians, engineers, and scientists to work with continuously changing quantities. The location of a car, driving at a constant speed, can be calculated easily with algebra (even before the use of GPS). The position of a falling body or the curving path of a planet orbiting the sun is constantly changed by the influence of a gravitational field — the problems are not so easy! The key idea is to break the problem into small linear pieces, then let those small pieces become vanishingly small until they, well, vanish.

Calculus was developed some 400 years ago as a formal branch of mathematics, and is a relatively new subject compared with algebra and trigonometry, many of whose techniques were worked out thousands of years ago. Still the need to work with changing quantities and infinitesimally small quantities was there even with the early mathematicians in ancient Greece.

Zeno of Elea was a royal pain (both literally and figuratively) to the ancient Greeks in 450 BCE. He bedeviled the philosophers of that time (and later) with a series of paradoxes in the form of thought problems. The problems looked perfectly obvious, but when Zeno argued with perfect logic and got a conclusion that was exactly opposite of what was perfectly obvious, the wise and learned of the time tended to get upset.

One of Zeno's most famous paradoxes concerned a race between Achilles and a tortoise. Achilles was a legendary hero of the Trojan War (read Homer's *Iliad*) who was renowned for his athletic prowess. Zeno would ask his latest victim to imagine Achilles in a race against a tortoise. Being confident of his ability, Achilles invited the tortoise to take a head start. (This is different than *The Tortoise and the Hare*, trust me. Achilles was serious about the race.)

Achilles watched the tortoise plod off until it was half way to the finish line. Now he flashed into action racing after the tortoise. It was here that Zeno pointed out a problem. He argued that Achilles could never catch the tortoise.

By the time Achilles reached the half-way point where the tortoise had been when he had started, the tortoise would have already plodded on farther towards the finish. By the time Achilles reached that new point, the tortoise would no longer be there, but would have moved on farther yet. Try as he might, by the time Achilles arrived at the point where the tortoise had been, the animal would have moved on even farther towards the finish.

Clearly, Zeno argued, since Achilles could never catch up to the tortoise, he could never pass the animal and win the race. At that point, the wise and learned philosophers of the day were usually screaming at Zeno. That is the nature of a paradox. We know Achilles would easily pass the tortoise and win the race, but following Zeno's seemingly perfectly logical arguments, he is forever stuck in second place.

The problem lies with those infinitesimal units of time as Achilles gets closer and closer to catching the tortoise. It took a couple of thousand years before mathematicians learned how to corral those infinitely tiny critters, but that doesn't mean that mathematicians weren't using the necessary idea before then — using it awkwardly and incompletely — but they were using it.

Archimedes, one of a number of mathematicians in ancient Greece, had a problem. Actually, he had (and solved) a whole lot of problems in mathematics and other areas of science. But, the one which concerns us is, how do you compute the area of a circle?

This was about 2300 years ago in ancient Greece. Archimedes wasn't able to "Google it" or even look it up the old fashioned way in a book because no one had ever figured out exactly how to do it yet. Oh, they could figure *about* what the area was with some rough and ready guidelines, but Archimedes wanted to know the exact figure.

The Greeks did know how to calculate the areas of a regular polygons (triangles, squares, hexagons, etc. with equal sides and equal angles) with any number of sides. Working with straight edges was okay. It was the curving (note: *continuously changing*) sides of a circle that they couldn't handle.

Archimedes used a method where the circle was captured between two regular polygons, one drawn with its sides just touching the outside the circle (the polygon *circumscribed* the circle), the other polygon drawn inside the circle with the vertices of the polygon just touching the circle (*inscribed*). Clearly the area of the circle was less than the circumscribed polygon and greater than the inscribed.

There are ways (and Archimedes knew them) to calculate the lengths of the sides of the regular polygons and use the lengths of the sides to calculate the area. The really clever part of Archimedes thinking was to pinch the circle between the two polygons (You will see this idea pop up several times in Calculus). Using polygons with more and more sides, he got polygons that were closer and closer in shape to an actual circle — still with the outside polygon a little bigger, the inside one a little smaller. With more sides, the polygons were closer to perfect circles, and Archimedes could calculate the actual area of the circle to any accuracy he wanted.

Well, to any accuracy Archimedes had the time and energy to calculate himself using the Greek's clumsy version of arithmetic. With this same idea, Archimedes calculated an approximate a value for pi. Using a polygon with 96 sides (a 96-gon), he calculated the actual value of pi was in the interval:

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

While the Greeks were on the right track to develop calculus with those ideas about infinitely small pieces of time and infinitely many-sided polygons, it wasn't until almost 2000 years later that the job got done. A big problem which had to be overcome was the development of arithmetic calculation techniques and an algebraic notation system which allowed the mathematics to be written concisely. (As an example, before William Jones in 1706 first designated pi with the symbol π, a mathematician would have to write, "The quantity which, when the diameter is multiplied by it, gives the circumference"— in Latin. Mathematicians could abbreviate, of course, but it was still clumsy.)

Around 1600, the scientists Galileo, Kepler, and Torricelli used techniques similar to Archimedes' methods to tackle problems in astronomy and physics. (Archimedes' writing on this were lost in antiquity and only rediscovered in the last 50 years.) Scientists at that time were specifically trying to solve problems of accelerated motion to understand how the solar system worked. Other mathematicians of the time solved specific problems and developed specific techniques which built on these ideas. Finally Isaac Newton (English) and Gottfried Leibniz (German) independently developed different, but mathematically equivalent, comprehensive and systematic approaches which we would come call Calculus. They didn't invent the subject out of nothing.

Rather, they generalized ideas and individual techniques into a unified approach which was easier to use and could be applied to many new areas.

Newton started working out his ideas by himself around 1665 but didn't publish them until 1704. Leibniz published his work on calculus in 1684. While Newton usually gets most of the credit, at least in the English speaking world, the Calculus you will learn follows the approach and notation system which Leibniz used.

The Calculus of Newton and Leibniz freely used the ideas of *infinitesimals* and *infinity*, concepts which made many mathematicians uncomfortable. (Remember the trouble Zeno caused using such ideas?) The idea that an infinite number of points, a location on the number line with a size equal to **zero,** actually adding up to something (a line) was slippery and difficult to tie down firmly mathematically. This concept of the infinitesimal caused political and theological problems (as did so many other new ideas) during the Renaissance period in western Europe.

Understanding infinity relates to the continuity problem on the number line. Between any two numbers on the number line, there are an infinite number of other numbers. Yet those individual points on the line representing numbers have no size (length = 0), but, when taken all together, form a **continuous** line with no holes or gaps.

The problem with infinity and continuity bothered mathematicians because there was no solid, theoretical grounds for proving any of this in a mathematically acceptable proof. However, the Calculus gave mathematicians powerful new tools to attack a wide range of problems, and they weren't about to give up their new toy just because a group of stuffy nit-pickers could not (literally) connect all the dots.

It took more time, but theoretical mathematicians finally caught up in the early 1800s when Augustin Louis Cauchy (French) and others developed the definitions and technique of proof you will learn as the delta-epsilon method. This finally put Calculus on a rigorous theoretical foundation. During this period Calculus was also extended by applying it to multiple dimensions, complex numbers, and other mathematical structures.